ON A PROBLEM OF KLEE CONCERNING CONVEX POLYTOPES

BY

P. McMULLEN

ABSTRACT

A non-simplicial d-polytope is shown to have strictly fewer k-faces ($[(d-1)/2] \leqq k \leqq d-1$) then some simplicial d-polytope with the same number of vertices; actual numerical bounds are given. This provides a strong affirmative answer to a problem of Klee.

Let P be a d-polytope, and for $0 \leqq k < d$, let $f_k(P)$ be the number of its *k-faces*. (We shall follow the terminology of [1] throughout the paper.) We denote by $\mu_k(v,d)$ the maximum possible number of k-faces of a d-polytope with v vertices:

$$\mu_k(v,d) = \max\{f_k(P) \,|\, P \text{ a } d\text{-polytope, } f_0(P) = v\}.$$

If we let $f_k(v,d)$ be the number of k-faces of the cyclic polytope $C(v,d)$ ([1, §4.7], see also [8]), or of any simplicial neighbourly d-polytope with v vertices ([1, §7.2 and §9.6]), then the Upper-bound Conjecture states that for all $v > d > k \geqq 1$, $\mu_k(v,d) = f_k(v,d)$. The Upper-bound Conjecture has been proved in almost all cases, but it still remains open for certain numbers of vertices in dimensions greater than 8. (For further details, see [1, 2, 3, 5].)

Quite apart from the problem of determining $\mu_k(v,d)$, it is obviously of interest to consider the problem of determining the maximum possible numbers of faces of polytopes in certain restricted classes. For example, centrally symmetric polytopes have been investigated in [1, §6.4, 3, 6], and the results of [3, 6] in particular show that the upper-bound problem in this case is probably very difficult. In this paper, we shall find a strong affirmative answer to a problem of Klee, who asked ([4]) if it were true that if, for some $[(d-1)/2] \leqq k \leqq d-1$, a d-polytope P with v vertices had $\mu_k(v,d)$ k-faces, then P was simplicial. We shall show

THEOREM. *Let P be a non-simplicial d-polytope with v vertices. Then for* $[(d-1)/2] \leqq k \leqq d-2$,

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$$f_k(P) \leqq \mu_k(v,d) - \binom{[d/2]+1}{k-[(d-1)/2]},$$

and

$$f_{d-1}(P) \leqq \mu_{d-1}(v,d) - [d/2].$$

Grünbaum ([1, addendum]) reports a private communication from Perles answering Klee's original question (although no proof has been published), and the case $k = d - 1$ of Klee's problem is answered by [1, theorem 10.1.1]. The theorem stated above does not involve the particular value of $\mu_k(v,d)$, although we shall show that, if the Upper-bound Conjecture holds, then the inequality of the theorem is sharp.

The proof of the theorem depends upon appropriate modifications of the procedures of pulling and pushing the vertices of a polytope, described in [1, §5.2; 4]. It is known that if we pull (or push) each of the vertices of a polytope in turn, we change it into a simplicial polytope with the same number of vertices, and a least as many faces of higher dimension. We shall show that we can assume, without loss of generality, that P has a simplicial facet F which is not a simplex, and that by pulling some vertex of F, for $[(d-1)/2] \leqq k \leqq d-1$, we can increase $f_k(P)$ by at least the deficit given in the statement of the theorem.

The first step in the proof is to show that we can assume that P has a simplical facet F with $d+1$ vertices. First suppose that P has a facet F which is not a simplex, so that F has at least $d+1$ vertices. F determines a supporting hyperplane $H = \operatorname{aff} F$ of P. We now pull, in H, each of the vertices of F in turn. That is, if x is a vertex of F, then we replace x by a new vertex x' (again in H) which is beyond each facet of P which contains x, except F (and so beyond each facet of F which contains x), and beneath every other facet of P (and so beneath every facet of F which does not contain x). The resulting $(d-1)$-polytope F' is a facet of the new polytope P', and an easy modification of [1, theorem 5.2.3] shows that

LEMMA. *For each k with $1 \leqq k \leqq d-1$, $f_k(P') \geqq f_k(P)$.*

As a result, we have not decreased the number of k-faces, and we still have a non-simplicial polytope. We now relabel P', F' as P, F, respectively. If F has more than $d+1$ vertices, then we choose an arbitrary subset of $d+1$ vertices of F (which by our assumption about F we can assume to be in general position in $H = \operatorname{aff} F$). We now push the remaining vertices of F in turn; that is, if x is one of these vertices, we replace it by a new vertex $x' \in \operatorname{int} P$ (the interior of P), such that the half-open line segment $[x', x]$ meets no hyperplane spanned by vertices of P. If we call the resulting polytope P', then again the lemma holds.

So, the facet F can be assumed to be a simplicial $(d-1)$-polytope with $d+1$ vertices, and so F is a polytope of type $T^{r,s}$ (say), where $1 \leqq r \leqq s$, $r+s=d-1$ ([7, §3.4]; Grünbaum denotes this polytope by T_r^d or T_s^d in [1, §6.1]). We can denote the vertices of $T^{r,s}$ by $y_0, \cdots, y_r$, $z_0, \cdots$, z_s, where $\operatorname{conv}\{y_0, \cdots, y_r\}$ and $\operatorname{conv}\{z_0, \cdots, z_s\}$ are r-and s-simplices, respectively, which meet in a single relatively interior point of each.

A k-face of $T^{r,s}$ $(k < d-1)$ is determined by any subset of $t \leqq r$ vertices from $\{y_0, \cdots, y_r\}$ and $u \leqq s$ from $\{z_0, \cdots, z_s\}$, with $t+u=k+1$. So, the number of k-faces of $T^{r,s}$ which contain y_0 is

$$\sum_{t+u=k,\, t<r,\, u\leqq s} \binom{r}{t}\binom{s+1}{u}.$$

To each of these k-faces corresponds exactly one $(k-1)$-face of $T^{r,s}$ which does not contain y_0. However, the total number of $(k-1)$-faces of $T^{r,s}$ $(k \leqq d-1)$ which do not contain y_0 is

$$\sum_{t+u=k,\, t\leqq r,\, u\leqq s} \binom{r}{t}\binom{s+1}{u},$$

and so the total number of $(k-1)$-faces G of $T^{r,s}$ such that $\operatorname{conv}\{(y_0\} \cup G)$ is not a k-face of $T^{r,s}$ is

$$\binom{s+1}{k-r}.$$

Since $r+s=d-1$, $r \leqq s$ and $k \geqq [(d-1)/2]$, this number is strictly positive, and clearly takes its minimal value

$$\binom{[d/2]+1}{k-[(d-1)/2]}$$

when $r = [(d-1)/2]$ and $s = [d/2]$. We now see at once that if we pull the vertex y_0 of P, we must increase the number of k-faces of P by at least this number (or this number less 1 if $k = d-1$), which proves the theorem.

We shall conclude by giving a brief description of a polytope which achieves the bound of the theorem, under the assumption that the Upper-bound Conjecture holds. Let $C(v,d)$ be the cyclic polytope, whose vertices are, in order along the moment curve ([1, §4.7]), $x_1, x_2, \cdots, x_v$. Now $\text{conv}\{x_1, \cdots, x_{v-1}\}$ is the cyclic polytope $C(v-1,d)$, and x_v is beyond certain certain facets of $C(v-1,d)$ (precisely those which are not facets of $C(v,d)$), and beneath the others. If d is even, let $F = \text{conv}\{x_1, x_{v-d+1}, \cdots, x_{v-1}\}$, and if d is odd, let $F = \text{conv}\ \{x_{v-d}, \cdots x_{v-1}\}$ so that x_v is beyond F. Let y be a point of the hyperplane aff F, such that y is beneath or beyond exactly the same facets of $C(v-1,d)$ (except F) as x_v, and let $P = \text{conv}(C(v-1,d) \cup \{y\})$. (The only difficult part of this example is to demonmonstrate that such a point y exists; this may be seen by considering the facets of $C(v-1,d)$ adjacent to F.) Then the facet $\text{conv}(F \cup \{y\})$ of P is of type $T^{r s}$ ($r = [(d-1)/2]$, $s = [d/2]$), and the vertex y_0 in the last part of the proof of the theorem can be taken to be y. If we pull the vertex y of P to x_v, we recover $C(v,d)$, and the only change in the number of k-faces arises from the facet $T^{r,s} = \text{conv}(F \cup \{y\})$. That is, P achieves the upper-bound of the theorem.

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MICHIGAN STATE UNIVERSITY,
EAST LANSING
AND
WESTERN WASHINGTON STATE COLLEGE,
BELLINGHAM, WA.